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The Pólya Algorithm on Convex Sets

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It is shown that there exist closed, convex sets in R^n for which the best p-norm approximations to a fixed element of R^n fail to converge as $p \to \infty$. Furthermore, it is shown that even if the best approximations converge, they need not converge to the strict best uniform approximation. © 1989 Academic Press, Inc.

INTRODUCTION

For $x \in R^n$ we define the p-norm of x, $1 \leqslant p < \infty$, by $\|x\|_p = (\sum |x_i|^p)^{1/p}$ and define $\|x\|_\infty$, the uniform norm of x, as max $|x_i|$. Denote R^n with the p-norm by l^p. If M is closed and convex in R^n, not containing y, then $z \in M$ is said to be a best approximation from M to y with respect to a norm $\| \;\|$, if $\|z - y\| = \min_{s \in M} \|s - y\|$. Since we may translate M, henceforth we will assume that $y = 0$.

In this setting, best approximations must exist for any norm, and they are unique for the p-norms, $1 < p < \infty$. Denote by x_p the p-norm best approximation to 0 from M. Although for $p = \infty$ there may be more than one best approximation, there is a distinguished best uniform approximation known as the strict (best uniform) approximation. This element is constructed as follows. For M as above, let W be the set of best l^∞ approximations. For each $z \in W$ let $|z|$ be the vector whose coordinates are given by the values $|z_i|$ arranged in non-decreasing order. Impose the lexicographic ordering on the vectors $|z|$. There exists a unique $w \in W$ which has $|w|$ minimal in this ordering [8]. This element is defined to be the strict uniform best approximation.

If we denote by $x_p = x_p(M)$ the l^p best approximation to 0 for $1 < p < \infty$, then for any sequence $p_i \to \infty$, x_{p_i} must contain a convergent subsequence. If x is a limit of such a subsequence, then x must be a best uniform approximation. In general, the net $\{x_p : p > 1\}$ may have many limit points. If this net has a limit as $p \to \infty$, we say that the Pólya algorithm converges.

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Pólya [7] first considered the analogous problem in polynomial approximation on an interval. Questions regarding the convergence of the net $\{x_p : p > 1\}$ in R^n first arose in the contexts of solutions of overdetermined systems of equations and discrete polynomial subspace approximation [4]. Cheney and Curtis [1] first posed the question of whether the Pólya algorithm must converge when the approximating set is an affine subspace of R^n. This was answered in the affirmative by Descloux [2] who showed that the net converges to the strict approximation. This is in strong contrast to the corresponding problem in $L^p[0, 1]$, where the Pólya algorithm may fail to converge even when the approximating set is a one-dimensional affine subspace [2]. Attempts have been made to generalize Descloux's results to a wider class of approximating sets. Houtari *et al.* [5] showed that the Pólya algorithm must converge to the strict approximation whenever the convex approximating set satisfies a condition they termed E-cylindrical. It remained open whether this condition was necessary, or whether the results in [5] could be generalized to arbitrary convex sets in R^n. The following examples show that these results are in some sense sharp and to not generalize.

The first example shows that the Pólya algorithm may converge to a point other than the strict approximation.

EXAMPLE 1. Let $K \subset R^3$ be the region enclosed by the parabola $x_1 = (x_2 - 3)^2 + 3$ in the plane $x_3 = 1$, i.e.,

$$K = \{(x_1, x_2, x_3): 3 \leqslant x_1 \leqslant 6,\ 3 + \varepsilon \geqslant x_2 \geqslant 3 - \varepsilon,\ x_3 = 1,\ \varepsilon = \sqrt{x_1 - 3}\}.$$

For $1 < p < \infty$, let $\xi = \xi_p$ be the best l^p approximation from K to 0. Since ξ must lie on the lower boundary of K, $\xi = (3 + \delta^2,\ 3 - \delta,\ 1)$ for some $\delta = \delta_p \geqslant 0$. Let H be the convex hull of K and $(3, 3, 0)$. The set H is closed and convex and the strict approximation from H to 0 is $w = (3, 3, 0)$. Let x_p be the best l^p approximation from H to 0. We will show that x_p does not converge to w.

For each x in H, let $\varphi_p(x) = \|x\|_p^p$. For x on the boundary of K, $x = (3 + s^2,\ 3 - s,\ 1)$, so φ_p can be considered as a function of s. Since $\varphi_p(s) = (3 + s^2)^p + (3 - s)^p + 1$, $\varphi_p''(s) > 0$ for $s \in (0, 3)$ and $p > 2$. In order to estimate the location of the minimum of φ_p, we now compute $\varphi_p(p^{-a})$ for various values of a. We have that

$$\varphi_p(p^{-a}) = 3^p\{[1 + 3^{-1}p^{-2a}]^{p^{2a}p^{1-2a}} + [1 - 3^{-1}p^{-a}]^{p^a p^{1-a}}\} + 1.$$

In particular, for $a = 1$ we have

$$\varphi_p(1/p) = 3^p\{[(1 + 3^{-1}p^{-2})^{p^2}]^{1/p} + [1 - 3^{-1}p^{-1}]^p\} + 1$$

which is asymptotic to $3^p(1+e^{-1/3})$ as $p \to \infty$. Similar computations show that $\varphi_p(p^{-1/2})$ is asymptotic to $3^p(e^{1/3})$ and that $\varphi_p(p^{-1/4})$ is larger than $2(3^p)$ for large p. Since $\varphi_p(s)$ is concave up, we must have that, for large p, the minimal value for $\varphi_p(s)$ must occur between p^{-1} and $p^{-1/4}$, i.e., $p^{-1} \leqslant \delta \leqslant p^{-1/4}$.

Note that x_p, the best l^p approximation from H to 0, must be of the form $x_p = \lambda w + (1-\lambda)z$, where $w = (3, 3, 0)$ and $z \in K$. Furthermore, since x_p is optimal, z must lie on the boundary of K. Hence $z = (3+\alpha^2, 3-\alpha, 1)$ and $x_p = (3+(1-\lambda)\alpha^2, 3-(1-\lambda)\alpha, 1-\lambda)$ for some $\alpha = \alpha_p \geqslant 0$ and $\lambda = \lambda_p \in [0, 1]$. Let $\eta_p = (3+(1-\lambda)\alpha^2, 3-(1-\lambda)\alpha, 1)$. Since $\eta_p \in K$ and it agrees with x_p in the first and second coordinates, we must have

$$\varphi_p(x_p) + 1 \geqslant \varphi_p(\eta_p) \geqslant \varphi_p(x_p)$$

and

$$\varphi_p(\xi_p) + 1 \geqslant \varphi_p(\eta_p) \geqslant \varphi_p(\xi_p).$$

Let $\psi_p = (3+\beta^2, 3-\beta, 1)$, where $\beta = \beta_p = \sqrt{1-\lambda_p}\,\alpha_p$. Then ψ_p is in the boundary of K and it agrees with η_p in the first and third coordinates. We must have $\varphi_p(\xi_p) + 1 \geqslant \varphi_p(\psi_p) \geqslant \varphi_p(\xi_p)$ and, equivalently, $\varphi_p(\delta) + 1 \geqslant \varphi_p(\beta) \geqslant \varphi_p(\delta)$. Furthermore, since for large p $\varphi_p(p^{-1/4}) \gg \varphi_p(p^{-1/2})$ and $\varphi_p(p^{-1}) \gg \varphi_p(p^{-1/2})$, we know that $p^{-1/4} \geqslant \beta_p \geqslant p^{-1}$.

We now estimate the quantity $\Delta_p = \varphi_p(\eta_p) - \varphi_p(\psi_p)$. If $\Delta_p > 1$, then $\varphi_p(\psi_p) < \varphi_p(x_p)$, which is impossible. Thus $1 \geqslant \Delta_p$. Note that

$$\begin{aligned}
\Delta_p &= (3-(1-\lambda)\alpha)^p - (3-\sqrt{1-\lambda}\,\alpha)^p \\
&= (3-\sqrt{1-\lambda}\,\beta)^p - (3-\beta)^p \\
&= (3-\beta+(1\sqrt{1-\lambda})\beta)^p - (3-\beta)^p \\
&\geqslant (3-\beta)^p + p + p(3-\beta)^{p-1}(1-\sqrt{1-\lambda})\beta - (3-\beta)^p.
\end{aligned}$$

Since $\beta > p^{-1}$, we have

$$1 \geqslant \Delta_p \geqslant (1-\sqrt{1-\lambda_p})(3-\beta_p)^{p-1},$$

so it must be that $\lambda_p \to 0$ as $p \to \infty$. Since $\beta_p \geqslant p^{-1/4}$, $\beta_p \to 0$ and hence, $\alpha_p \to 0$ as $p \to \infty$. Thus

$$x_p = (3+(1-\lambda_p)\,\alpha_p^2, 3-(1-\lambda_p)\alpha_p, 1-\lambda_p) \to (3, 3, 1)$$

as $p \to \infty$.

In our discussion of Example 2, we will use the following observations concerning perturbations of an approximating set. Let Γ consist of all compact convex subsets of R^3. For any set C in Γ, let

$$U(C, \varepsilon) = \bigcup_{x \in C} \{y: \|y - x\|_\infty < \varepsilon\}$$

and, for A and B in Γ, let

$$D(A, B) = \inf\{\varepsilon: A \subset U(B, \varepsilon) \text{ and } B \subset U(A, \varepsilon)\},$$

the Hausdorff distance between A and B. It is easily shown that

(1) If $1 < p < \infty$ and if $F: \langle \Gamma, D \rangle \to \langle R^3, \|\cdot\|_\infty \rangle$ is defined by $F(M) = x_p(M)$, then F is continuous, and

(2) If $1 < p < \infty$, if $M \in \Gamma$, and if M' is the closed convex hull of any subset of M which contains $x_p(M)$, then $x_p(M') = x_p(M)$.

EXAMPLE 2. Let H be the set given in Example 1. We now construct a set, H', which is the limit of a sequence of polytopes in H, and on which the Pólya algorithm does not converge.

Let $\{p_n\}$ be any strictly increasing sequence such that $p_1 \geq 1$ and $p_n \to \infty$. Then $x_{p_n}(H) \to (3, 3, 1)$ as $n \to \infty$. Let $q_1 = p_1$ and let H_0 be the convex hull of the points $(3, 3, 0)$, $(3, 3, 1)$ and $x_{p_1}(H)$. Since H_0 is a polytope and the strict approximation to 0 in H_0 is $(3, 3, 0)$, the results in [5] show that there exists a real number $q_2 > 2q_1$ such that $x_{q_2}(H_0)$ has third coordinate less than $\frac{1}{2}$. By (1) and (2), there exists q_3 in the set $\{p_n: n > 1\}$ such that $q_3 > 2q_2$ and, if H_1 is the convex hull of $H_0 \cup \{x_{q_3}(H)\}$, then $x_{q_2}(H_1)$ has third coordinate less than $\frac{1}{2}$ and $x_{q_3}(H_1) = x_{q_3}(H)$.

Continuing in this fashion, we obtain a sequence of polytopes $H_0, H_1, \ldots$, such that, for each $k \geq 0$, H_k is the convex hull of $H_{k-1} \cup \{x_q(H)\}$, where $q = q_{k+2} \in \{p_n: n = 1, 2, \ldots\}$, $x_{q_{k+1}}(H)$ has third coordinate less than $\frac{1}{2}$, and $x_{q_{k+2}}(H_k) = x_{q_{k+2}}(H)$. Let $H' = \bigcup_{n=0}^{\infty} H_k$. Then the sequence $\{x_{q_k}(H')\}$ does not converge.

Remarks. Example 2 is sharp in two senses. First, since the Pólya algorithm converges in R^2 for every closed, convex approximating set [5], the dimension in this example is minimal. Second, since the algorithm is known to converge for M any polytope in R^n [5], and since H' is an (essentially) disjoint union of tetrahedra, there is little possibility of ensuring convergence on more general sets.

Example 2 can also be viewed as a generalization of results in [3], where it was shown that even if x_p converges, it need not converge monotonically.

Finally, we contrast the examples above with the results of Landers and Rogge. It is an immediate consequence of [6] that the Pólyaesque limit of the x_p, $p \to 1$, must converge to the "natural" best l^1 approximation.

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